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THE VARIATIONAL INEQUALITY FORMULATION FOR THE DEFORMATION THEORY IN PLASTICITY AND ITS NON-ITERATIVE SOLUTION

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Abstract

In this paper, the deformation theory in plasticity is formulated in the variational inequality, which can relax the constraint conditions of the constitutive equations. The new form makes the calculation more convenient than general energy forms and have reliable mathematical basis. Thus the plasticity theory may be solved by means of the quadratic programming instead of the iterative methods. And the solutions can be made in one step without any diversion of the load.

Key words deformation theory in plasticity, variational inequality, quadratic programming

In an elastoplastic problem the constitutive equation is an inequality. Under the classical variational principles these problems can hardly be solved, because in the classical variational problems the arguments are not constrained in their domain of definition.

One of the approaches is the direct application of the programming method, in which the extremum of the functional is studied by means of the linearization of the nonlinear problem to a series of the linear complimentary problems.

Now in this paper we will establish a variational inequality formulation for the deformation theory of plasticity and find a satisfactory non-iterative solution for the deformation theory in plasticity by using the quadratic programming method of variational inequality.

I. Deformation Theory in Plasticity

Suppose that the simple loading condition for the material is satisfied. So the relation between the compliments of the stress tensor should be kept unchangeable as follows:

$$\sigma_{ij}=\sigma^{\circ}_{ij}\,\Phi,\quad S_{ij}=S^{\circ}_{ij}\,\Phi \tag{1.1}$$

where Φ is a parameter of proportion σ°_{ij}, S°_{ij} are non-zero stress tensor at a moment of time. According to the associate flow law, the increment of the plastic strain and the equivalent strain can be written as[2,3]

$$d\varepsilon_{ij}^{p}=\frac{3S_{ij}}{2\bar{\sigma}}d\lambda \tag{1.2}$$

$$d\bar{\varepsilon}^{p}=\sqrt{(2/3)d\varepsilon_{ij}^{p}\cdot d\varepsilon_{ij}^{p}}=d\lambda \tag{1.3}$$

Note

$$R_{ij}=\frac{3}{2}\ \frac{S_{ij}^{\circ}}{\bar{\sigma}^{\circ}}=\frac{3}{2}\ \frac{S_{ij}}{\bar{\sigma}}$$

So we have Eq. (1.3) in the form of (1.4).

$$d\varepsilon_{ij}^{p}=R_{ij}d\lambda \tag{1.4}$$

And, in case of the simple loading, ε_{ij}^{p} , $\bar{\varepsilon}^{p}$ will be

$$\varepsilon_{ij}^{p}=\int d\varepsilon_{ij}^{p}=\int R_{ij}d\lambda=R_{ij}\lambda \tag{1.5}$$

$$\bar{\varepsilon}^{p}=\int d\bar{\varepsilon}^{p}=\int d\lambda=\lambda \tag{1.6}$$

Again by integration of

$$d\sigma_{ij}=D_{ijkl}\cdot\ d\varepsilon_{kl}^{e}=D_{ijkl}(d\varepsilon_{kl}-d\varepsilon_{kl}^{p}) \tag{1.7}$$

we may have

$$\sigma_{ij}=D_{ijkl}(\varepsilon_{kl}-R_{kl}\cdot\lambda) \tag{1.8}$$

and the equivalent stress

$$\begin{aligned}\bar{\sigma}&=\sqrt{\frac{3}{2}S_{ij}\cdot S_{ij}}=\frac{\sqrt{3/2}\,S_{ij}\cdot S_{ij}}{\sqrt{S_{ij}\cdot S_{ij}}}=\frac{\sqrt{3/2}\,S_{ij}\cdot\sigma_{ij}}{\sqrt{S_{ij}\cdot S_{ij}}}\\&=R_{ij}\sigma_{ij}=w_{kl}\cdot\varepsilon_{kl}-\tilde{D}\lambda\end{aligned} \tag{1.9}$$

where

$$w_{kl}=R_{ij}\cdot D_{ijkl},\quad \tilde{D}=R_{ij}D_{ijkl}R_{kl}$$

So the equivalent stress $\bar{\sigma}$ is discribed in a linear function of σ_{ij} , ε_{ij} and λ .
Here the Mises yield postulate for material is supposed to be hold,

$$f=\bar{\sigma}-\sigma_{s}-h\left(\int d\bar{\varepsilon}^{p}\right)\leqslant 0 \tag{1.10}$$

or

$$w_{ij}\varepsilon_{ij}-\sigma_{s}-\tilde{D}\lambda-h(\lambda)\leqslant 0 \tag{1.11}$$

Because in a simple loading process, only the equal-axial hardening is considered, the hardening function can be linearized as[2]

$$h(\lambda)=\tilde{h}\cdot\lambda$$

and thus Eq. (1.11) can be written as

$$f=w_{ij}\varepsilon_{ij}-\sigma_{s}-(\tilde{D}+\tilde{h})\lambda\leqslant 0 \tag{1.12}$$

Where λ is the vector of flow parameter, and its compliments λ_{a} satisfy the following relations.

$$\lambda_a \begin{cases} \geqslant 0, & \text{when } f_a = 0 \\ = 0, & \text{when } f_a < 0 \end{cases} \tag{1.13}$$

$a=1,2,\cdots, m$ (m —number of surfaces of the plastic potential)

So the linearized yield condition for strains e_{ij} of the total deformation is:

$$(\bar{D}+\bar{h})\lambda - w^T e(u) + \sigma_s \geqslant 0 \tag{1.14}$$

$$\lambda_a[(\bar{D}+\bar{h})\lambda - w^T e(u) + \sigma_s] = 0 \tag{1.15}$$

In the deformation theory, the equilibrium Eq., boundary conditions and the continuity Eq. are:

$$\sigma_{ij,j} + f_i = 0 \tag{1.16}$$

$$\sigma_{ij} \cdot n_j = p_i, \quad \text{on } \Gamma_p \tag{1.17}$$

$$u_i = u_i^\circ, \quad \text{on } \Gamma_u \tag{1.18}$$

$$e_{ij} = (u_{i,j} + u_{j,i})/2 \tag{1.19}$$

Finally, we obtain the whole set of Eqs. (1.14)–(1.19).

II. Equivalent Variational Inequality Formulation

For the deformation theory of plasticity in following discussion the displacements $u_i (i=1,2,3)$ will be taken as the state variables of the system, where the flow parameter λ as the control variable.

Using the following definition for space:

$H_1^1(\Omega) = \{u \mid u \in H_1(\Omega),\ u|_{\Gamma_u} = u^\circ\}$, $H_1(\Omega)$——Sobolev Space

$H_1^0(\Omega) = \{u \mid u \in H_1(\Omega),\ u|_{\Gamma_u} = 0\}$

$\bar{H}_1^1(\Omega) = [H_1^1(\Omega)]^3$, $\bar{H}_1^0(\Omega) = [H_1^0(\Omega)]^3$

$\bar{L}_2(\Omega) = [L_2(\Omega)]^m$, $L_2(\Omega)$—— Hilbert Space

$\bar{K} = \{\{u,\lambda\} \mid \{u,\lambda\} \in \bar{H}_1^1(\Omega) \times \bar{L}_2(\Omega), \lambda_k \geqslant 0, k=1,2,\cdots,m\}$

We obtain the new form which equivalent on (1.14)–(1.19).

Find $\{u,\lambda\} \in \bar{K}$ which leads to

$$a(u,v-u) - b(v-u,\lambda) + c(\lambda, r-\lambda) - b(u, r-\lambda) + j(r-\lambda) \geqslant L(v-u) \quad \forall \{v,r\} \in \bar{K} \tag{2.1}$$

where

$$a(u,v) = \int_\Omega \varepsilon^T(u) D\varepsilon(v) d\Omega$$

$$b(u,\lambda) = \int_\Omega \sum_{a=1}^m \varepsilon^T(u) w \cdot \lambda_a d\Omega$$

$$c(\lambda, r) = \int_\Omega \sum_{a,j}^m \lambda_a (\bar{D}+\bar{h}) r_j d\Omega$$

$$j(r)=\int_{\Omega}\sum_{a}^{m} r_{a}\sigma_{a} d\Omega$$

$$L(v)=\int_{\Omega} v^{T} f d\Omega+\int_{\Gamma_{p}} v^{T} p d\Gamma$$

The equivalence (2.1) and (1.14)–(1.19) can be proved.

(1) Assume that $\{u,\lambda\}\in \overset{\circ}{H}{}_1^1(\Omega)\times L_2(\Omega)$ and satisfy (1.14)–(1.19).

According to the virtual work principle, from (1.16)–(1.19) we can get

$$\forall \omega\in \overset{\circ}{H}{}_1^0(\Omega),\int_{\Omega}\varepsilon^{T}(\omega)D[\varepsilon(u)-R\lambda]d\Omega=\int_{\Omega}\omega^{T} f d\Omega+\int_{\Gamma_p}\omega^{T} p d\Gamma \tag{2.2}$$

or:

$$a(u,\omega)-b(\omega,\lambda)=L(\omega), \qquad \forall \omega\in \overset{\circ}{H}{}_1^0(\Omega) \tag{2.3}$$

Set

$$v=u+\omega\in \overset{\circ}{H}{}_1^1(\Omega)$$

then (2.3) will be

$$a(u,v-u)-b(v-u,\lambda)=L(v-u) \tag{2.4}$$

from (1.14) and (1.15) we get

$$\int_{\Omega}\sum_{a}\Big(\sum_{j}(\bar{D}+\bar{h})\lambda_j+\varepsilon^{T}(u)w+\sigma_a\Big)(r_a-\lambda_a)\geqslant 0 \tag{2.5}$$

$$\forall r_a\geqslant 0 \qquad (a=1,2,3,\cdots,m)$$

or

$$c(\lambda,r-\lambda)+b(u,r-\lambda)+j(r-\lambda)\geqslant 0 \tag{2.6}$$

Adding (2.4) and (2.6), we get

$$a(u,v-u)-b(v-u,\lambda)+c(\lambda,r-\lambda)+b(u,r-\lambda)+j(r-\lambda)\geqslant L(v-u)$$

$$\forall \{v,r\}\in \tilde{K}$$

(2) Assume that form (2.1) holds for an arbitrary $\omega\in \overset{\circ}{H}{}_1^0(\Omega)$, take $\{v,r\}=\{u\pm\omega,\lambda\}\in\tilde{K}$, we get

$$a(u,\pm\omega)-b(\pm\omega,\lambda)\geqslant L(\pm\omega) \tag{2.7}$$

it follows

$$a(u,\omega)-b(\omega,\lambda)=L(\omega) \tag{2.8}$$

Thus we can get form (2.2) which can be easily proved to satisfy (1.16)–(1.19).

Eq. (2.8) yields

$$a(u,v-u)-b(v-u,\lambda)=L(v-u) \tag{2.9}$$

Substituting it into (2.1), we get

$$c(\lambda,r-\lambda)+b(u,r-\lambda)+j(r-\lambda)\geqslant 0 \tag{2.10}$$

Setting $r=2\lambda\in L_2(\Omega)$, $r=0\in L_2(\Omega)$, we get

$$c(\lambda,\lambda)+b(u,\lambda)+j(\lambda)=0 \tag{2.11}$$

Substitution (2.11) into (2.10), leads to

$$c(\lambda,r)+b(u,r)+j(r)\geqslant 0 \tag{2.12}$$

or

$$\int_\Omega \sum_a \Big(\sum_j (\tilde{D}+\tilde{h})\lambda_j+\varepsilon^T(u)w+\sigma_s\Big)r_a\geqslant 0 \tag{2.13}$$

$$\forall r\in L_2(\Omega),\quad r_k\geqslant 0\quad (k=1,2,\cdots,m)$$

Considering the arbitrariness of $r_k\geqslant 0$, we get

$$\sum_j (\tilde{D}+h)\lambda_j+\varepsilon^T(u)w+\sigma_s\geqslant 0 \tag{2.14}$$

and consequently, we obtain (1.14).
Also from (2.11) we may have

$$\int_\Omega \sum_a \lambda_a\Big(\sum_j (\tilde{D}+h)\lambda_j+\varepsilon^T(u)w+\sigma_s\Big)=0 \tag{2.15}$$

because of $\lambda_k\geqslant 0$ from Eqs. (2.14),(2.15) we can obtain

$$\lambda_a\Big(\sum_j (\tilde{D}+\tilde{h})\lambda_j+\varepsilon^T(u)w+\sigma_s\Big)=0 \tag{2.16}$$

or we may have (1.15), and finally the equivalence of (2.1) and (1.14)–(1.19) is proved.

III. Minimization Form of Potential Energy and the Analytic Example

It can also be easily proved that problem (2.1) is equivalent on the following minimization problem:

Find

$$J(u,\lambda)=\min_{\{v,r\}\in\tilde{K}}[J(v,r)] \tag{3.1}$$

where

$$J(v,r)=\frac{1}{2}a(v,v)+\frac{1}{2}c(r,r)-b(v,r)+j(r)-L(v)$$

because $\tilde{K}$ is a close convex and the objective function of this problem is a convex function, (3.1) is a convex programming.

Example 1 The pressure test of a specimen of elastoplastic material with cross section A (Fig. 1). The upper plate is connected with an elastic ring with stiffness K , the stress-strain relationship of the material is depicted in Fig. 2, where E_0 and E_1 are the modulus of elasticity in the elastic and hardening stage respectively. For strain hardening material $E_1>0$, $E_1=0$, for ideal plastic and $E_1<0$ for softening material.

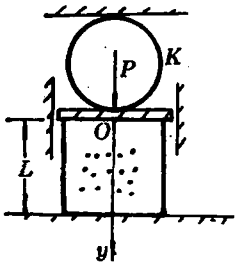

Fig. 1 Pressure test of a specimen

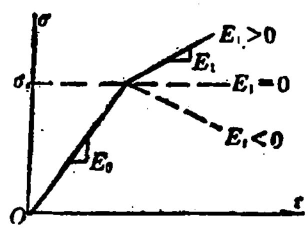

Fig. 2 Stress-strain relationship of the material

In this case $w_{ij}=E_0$, $\bar{D}=E_0$, and the material of specimen submits the Mises yield condition and the hardening rule of

$$dK=\frac{E_0E_1}{E_0-E_1}\lambda,\qquad \bar{h}\lambda=\frac{E_0E_1}{E_0-E_1}\lambda' \tag{3.2}$$

Thus we have

$$J(u,\lambda)=\int_0^L\int_A\frac{E_0}{2}\varepsilon^2dA\,dy-\int_0^L\int_A E_0\lambda\varepsilon dA\,dy-\left[Pv_0-\frac{1}{2}Kv_0^2\right]$$
$$+\frac{1}{2}\int_0^L\int_A\frac{E_0^2}{E_0-E_1}\lambda^2dA\,dy+\int_0^L\int_A\sigma_s\lambda dA\,dy \tag{3.3}$$

where

$$v_0=v\,|_{y=0}$$

Taking the function of displacement as $v=v(y)=\xi(L-y)$, which satisfies the condition of $v(L)=0$, we can write

$$J(u,\lambda)=\frac{1}{2}E_0\xi^2AL-E_0\xi\lambda AL-P\xi L+\frac{1}{2}KL^2\xi^2+\frac{1}{2}\,\frac{E_0^2}{E_0-E_1}\lambda^2AL$$
$$+\sigma_s\lambda AL \tag{3.4}$$

If the functional J takes its minimum value for $\lambda\geqslant0$, the Kuln-Tucker conditions should be satisfied.

$$\left.\begin{array}{ll}\partial J/\partial\xi=0 & \\ \partial J/\partial\lambda=0, & \text{when } \lambda>0\\ \partial J/\partial\lambda\geqslant0, & \text{when } \lambda=0\end{array}\right\} \tag{3.5}$$

Thus we obtain

$$\xi=\frac{E_0A\lambda+P}{KL+E_0A} \tag{3.6}$$

$$-\frac{E_0^2A\lambda+E_0P}{KL+E_0A}+\frac{E_0^2}{E_0-E_1}\lambda+\sigma_s=0,\quad \lambda>0 \tag{3.7}$$

$$-\frac{E_0^2A\lambda+E_0P}{KL+E_0A}+\frac{E_0^2}{E_0-E_1}\lambda+\sigma_s\geqslant0,\quad \lambda=0 \tag{3.8}$$

Introducing a relaxation variable $\nu\geqslant0$, Eqs. (3.7) and (3.8) may be rewritten as:

$$\left.\begin{array}{l} -\dfrac{E_0^2 A\lambda+E_0P}{KL+E_0A}+\dfrac{E_0^2}{E_0-E_1}\lambda+\sigma_s-\nu=0 \\ \nu\lambda=0,\ \nu\geqslant 0,\ \lambda\geqslant 0 \end{array}\right\} \tag{3.9}$$

From the complementarity of λ and ν we know that

$$\text{for } \lambda=0\ ,\quad \nu=\sigma_s-\frac{E_0P}{KL+E_0A}>0,\ \text{or } P<\sigma_s(A+KL/E_0),$$

$$\text{for } \nu=0\ ,\quad \lambda=\frac{(E_0-E_1)(E_0P-\sigma_s(E_0A+KL))}{E_0^2(E_1A+KL)}\geqslant 0.$$

And finally the solution of this problem can be found:

When $P<\sigma_s(A+KL/E_0)$, $\lambda=0$, the elastic solution is

$$\varepsilon=\xi=\frac{P}{KL+E_0A} \tag{3.10}$$

$$v=\xi(L-y)=\frac{P(L-y)}{KL+E_0A} \tag{3.11}$$

$$\sigma=E_0\varepsilon=\frac{PE_0}{KL+E_0A} \tag{3.12}$$

When $P\geqslant\sigma_s(A+KL/E_0)$, $\lambda>0$, the plastic solution is

$$\varepsilon=\xi=\frac{P-(1-E_1/E_0)A\sigma_s}{E_1A+KL} \tag{3.13}$$

$$v=\xi(L-y)=\frac{[P-(1-E_1/E_0)A\sigma_s][L-y]}{E_1A+KL} \tag{3.14}$$

$$\sigma=E(\varepsilon-\lambda\partial f/\partial\sigma)=\frac{E_0E_1P+KL(E_0-E_1)\sigma_s}{E_0^2(E_1A+KL)} \tag{3.15}$$

For softening materials ($E_1<0$), it is necessary that $K>-E_1A/L$, to ensure the uniqueness of the solution (or $\lambda>0$).

IV. Deformation Theory and Its FEM Solution

Now we will give a finite element solution for the piecewise linearized hardening problem, benefited from the above deduced variational inequality. The body Ω under investigation is divided into N_e elements, from which $N_1(N_1\leqslant N_e)$ are the elastoplastic element. It is suggested that every element may have only one state (elastic or plastic). The hardening curve of the e-th element consists of L_e $(L_e\geqslant 1)$ broken lines (Fig. 3), and then the number of the state equations is $L=\sum_{e=1}^{N_1} L_e$

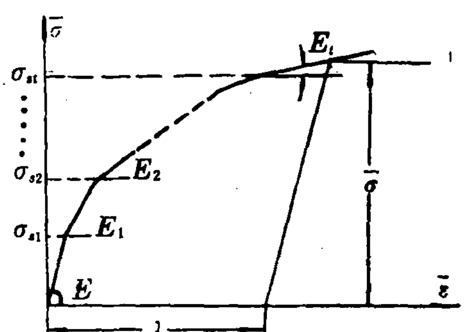

Fig. 3 The piecewise linearized hardening problem

Making a finite element discretization for Eq. (2.1), introducing the shape function N and operator B, interpolating u and ε by nodal displacements δ and v by nodal displacements φ :

$$u=N\delta,\ \ \varepsilon(u)=B\delta$$
$$v=N\varphi,\ \ \varepsilon(v-u)=B(\varphi-\delta)$$

we may obtain

$$[\varphi-\delta]^T[K\delta-C^T\lambda-t]+[r-\lambda]^T[U\lambda-C\delta+d]\geqslant 0 \tag{4.1}$$

where

$$K=\sum_{e=1}^{N_e}\int_{\Omega^e}B^T DB d\Omega$$

$$t=\sum_{e=1}^{N_e}\left\{\int_{\Omega^e}N^T f d\Omega+\int_{\Gamma^e}N^T P d\Gamma\right\}$$

$$C=\sum_{e=1}^{N_1}\int_{\Omega^e}\begin{bmatrix} w_x & w_y & w_z & w_{xy} & w_{yz} & w_{zx} \\ w_x & w_y & w_z & w_{xy} & w_{yz} & w_{zx} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_x & w_y & w_z & w_{xy} & w_{yz} & w_{zx} \end{bmatrix} B d\Omega$$

$$U=\sum_{e=1}^{N_1}\int_{\Omega^e}\begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1L_e} \\ m_{21} & m_{22} & \cdots & m_{2L_e} \\ \vdots & \vdots & & \\ m_{L_e 1} & m_{L_e 2} & \cdots & m_{L_e L_e} \end{bmatrix} d\Omega$$

$$m_{\alpha\beta}=\bar{D}+\frac{E_{\alpha-1}E_\alpha}{E_{\alpha-1}-E_\alpha}\delta_{\alpha\beta}$$

$$d=\sum_{e=1}^{N_1}\int_{\Omega^e}\{\sigma_{s1},\sigma_{s2},\cdots,\sigma_{sL_e}\}^T d\Omega$$

$$\lambda=\{\lambda_1^{(1)},\lambda_2^{(1)},\cdots\lambda_{L_1}^{(1)},\lambda_1^{(2)},\lambda_2^{(2)},\cdots,\lambda_{L_2}^{(2)},\cdots,\lambda_1^{(N_1)},\lambda_2^{(N_1)},\cdots,\lambda_{LN_1}^{(N_1)}\}^T$$

K and t are the elastic stiffness matrix and the load vector respectively. C and U are noted as constraint matrix and hardening matrix, and d as a vector related to the yield stress of material.

From (4.1), considering the arbitrariness of $\{\varphi,r\}$, we get

$$\begin{cases} K\delta-C^T\lambda-t=0 & (4.2) \\ U\lambda-C\delta+d\geqslant 0 & (4.3) \end{cases}$$

Involving a relaxation variable $\nu=\{\nu_1^{(1)},\nu_2^{(2)},\cdots,\nu_{L_1}^{(1)},\cdots,\nu_1^{(N_1)},\nu_2^{(N_1)},\cdots\nu_{LN_1}^{(N_1)}\}^T$ we have

$$\left.\begin{aligned} & K\delta-C^T\lambda-t=0 \\ & C\delta-U\lambda-d+\nu=0 \\ & \lambda^T\nu=0,\qquad \nu\geqslant 0,\ \lambda\geqslant 0 \end{aligned}\right\} \tag{4.4}$$

Eq. (4.4) discribes a quadratic programming problem with one free variable, and it has the same form as the system of equations[2] for the incremental theory of plasticity, which can be solved by the Lemke algorithm[6] for the linear complemental problem in quadratic programming. This approch makes it possible to solve the problem in one step without iteration.

Example 2 Consider a thick tube of ideal elastoplastic material with infinite length subjected to inner pressure (Fig 4)

The yield stress $\sigma_s=240$ MPa, the moduls of elasticity $E=200$ GPa, the inner radius $R_1=5$cm and the outer radius $R_2=15$cm.

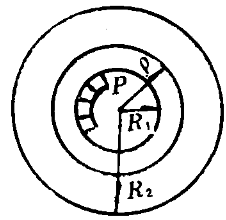

Fig. 4 Thick tube with infinite length

The tube is divided into 20 8 - node axisymmetric Serendipity elements. The result of computation by the presented scheme for the radius of elastoplastic interface is identical with that obtained from the exact solution.

Compared with the analytical solution, the results for displacements of the tube surfaces are given in Table 1, showing accuracy of the method.

Table 1 Radial displacement at $R=5$cm

P (MPa)		−20.000	−144.557	−174.620	−198.667	−233.022
ρ (cm)		Elastic region	6.000	7.000	8.000	10.000
$\mu=0.25$	Exact, solution	0.0007422	0.0058413	0.0082965	0.0112958	0.0188590
	Numerical result	0.0007464	0.0059182	0.0083141	0.0111862	0.0181073
$\mu=0.45$	Exact, solution	0.0008247	0.0063664	0.0087456	0.0115294	0.0182953
	Numerical result	0.0008296	0.0064703	0.0088072	0.0114894	0.0177198

Example 3 In order to make comparison we take the example of a rectangular specimen given in ref. [7] (see Fig. 5), subjected to the pressure of two rigid slabs. The interface condition between the slabs and the specimen are supposed to be fully rough contact, and the material of the specimen is considered as an ideal elastoplastic and subordinated to the Mises yield condition.

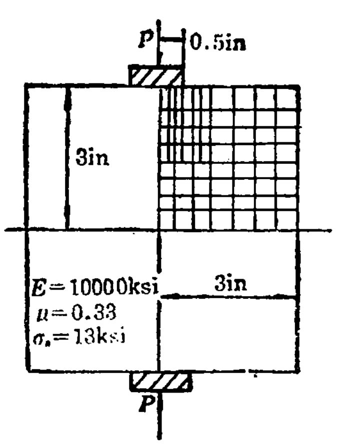

Fig. 5 Rectangular specimen subject to the pressure of two rigid slabs

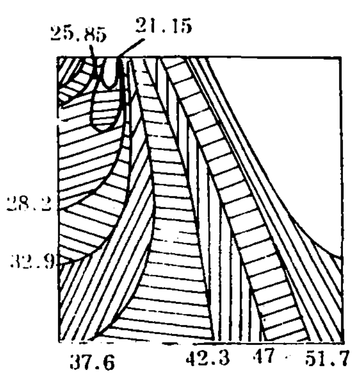

Fig. 6 Development of the plastic region

The discretization is made 1/4 region because of the symmetry of the specimen. Fig. 6 describes the development of the plastic region with increase of load P. It is seen that the elements near the corner of the rigid slabs enter into the plastic stage at first. Fig. 7 gives the P-δ relation. It is in good accordance with the results of ref. [7], where the author used 274 elements and had taken more than 120 load increments, while we used only 68 elements and 46 base exchanges. The present solution shows great advantage in accuracy and convergence.

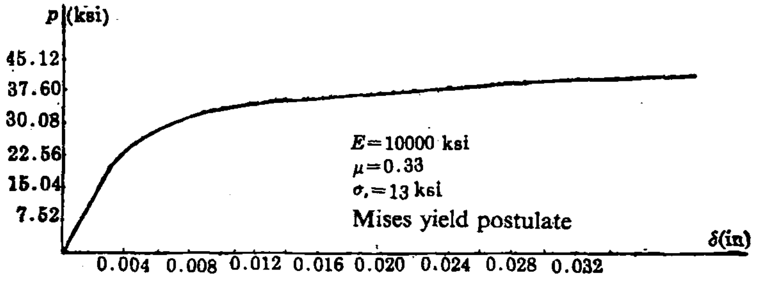

Fig. 7 P-δ Relation

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